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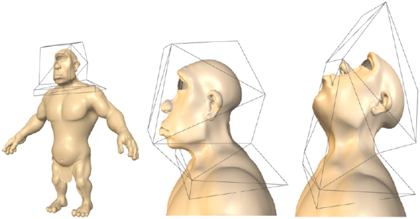

Green Coordinates

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Outline

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Contents

What are Barycentric Coordinates

- Idea: Spatial coordinates of a point are represented as linear combination of the vertices of an ambient cage.
- $x \in \mathbb{R}^d$ point, $v_i \in \mathbb{R}^d$ vertices of a cage P; find $\varphi_i(x)$ so that:

$$x = \sum_{i \in I_{\mathbb{V}}} \varphi_i(x) \cdot v_i$$

- Motivation:
  1. Interpolate function values given on the boundary:

$$f(x) := \sum_i \varphi_i(x) \cdot f(v_i)$$

  2. Move the cage vertices and see how the internal points move along:

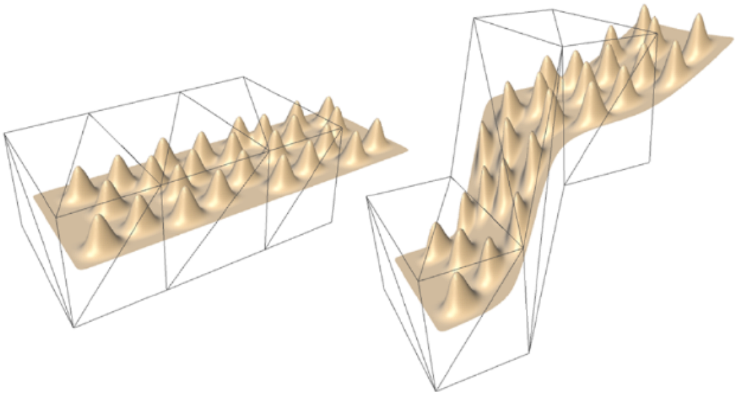
$$F(\cdot, P') : x \mapsto x' := \sum_i \varphi_i(x) \cdot v_i'$$

We look only at (2.) here; special case of (1.), with f being the transformation applied to P.

Problems with Existing Methods

- Linear combinations of cage vertices must lead to *affine-invariant* transformations, not *shape-preserving*.
- *Shape-preserving*
  - Close to rotations with isotropic scale
  - Infinitesimal circles are mapped to infinitesimal ellipsoids with bounded axis ratio (quasi-conformal)
- *Affine-invariant*
  - Affine transformation applied to cage results in same transformation applied to geometry
    $\Rightarrow$ problems with shearing and anisotropic scale
  - Especially: Changes in only one direction do not affect the other directions

Problems with Existing Methods



Original, affine-invariant transformation

Solution: Green Coordinates

Contents

Facts about Green Coordinates

- Paper from Y. Lipman, D. Levin, D. Cohen-Or, presented on SIGGRAPH 2008
- Can be used with piecewise smooth boundaries in any dimension
- Cages must not be necessarily simply connected
- Yields conformal transformations in 2D, quasi-conformal transformations in higher dimensions

Idea of Green Coordinates

- Take not only vertices of cage, but also face orientation (= normals) into account.
- P a cage, $v_i \in \mathbb{R}^d$ vertices ($i \in I_{\mathbb{V}}$), t_j faces with normals $n_j \in \mathbb{R}^d$ ($j \in I_{\mathbb{T}}$)

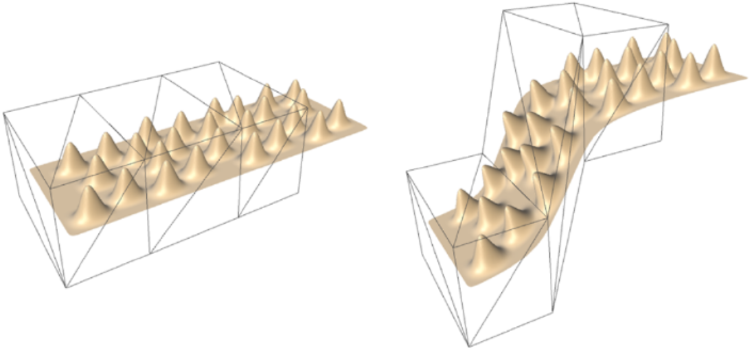
$$x = \sum_{i \in I_{\mathbb{V}}} \varphi_i(x) \cdot v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(x) \cdot n_j$$

- With cage change $P \mapsto P'$, transformation is then given by

$$F(\cdot, P') : x \mapsto x' = \sum_{i \in I_{\mathbb{V}}} \varphi_i(x) \cdot v'_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(x) \cdot s_j \cdot n'_j$$

- s_j scaling factors, chosen appropriately to obtain desired properties

Example Transformation



Original, transformation induced from Green Coordinates

Derivation of Green Coordinates

Theorem (Green's Third Identity)

Let $\Omega \subset \mathbb{R}^d$ with a smooth boundary, G_a a fundamental solution of the Laplace equation (i. e. $\Delta G_a(x) = \delta_{a,x}$). If $u : \Omega \to \mathbb{R}$ is twice continuously differentiable, then for all $a \in \Omega$, the following equality holds:

$$u(a) = \int_{\partial\Omega} \left(u(x) \cdot \frac{\partial G_a}{\partial n}(x) - G_a(x) \cdot \frac{\partial u}{\partial n}(x) \right) d\sigma_x + \underbrace{\int_{\Omega} G_a(x) \cdot \Delta u(x)\, dx}_{\text{vanishes if } u \text{ harmonic}}$$

Those functions G_a in $\mathbb{R}^d$ have the form

$$G_a(x) = \begin{cases} \frac{1}{2\pi} \log \|a - x\| & d = 2 \\ \frac{1}{(2-d)\omega_d} \|a - x\|^{2-d} & d \geq 3 \end{cases}$$

(with ω_d volume of the d-unit sphere).

Derivation of Green Coordinates

Treat coordinate functions $u = (x, y, z) : \Omega \to \mathbb{R}^3$ as special harmonic functions (in each component):

$$u(a) = a = \int_{\partial\Omega} \left(x \cdot \frac{\partial G_a}{\partial n}(x) - G_a(x) \cdot n(x) \right) d\sigma_x$$

Remark

Let $d = 2 \Rightarrow G_a(x) = \frac{1}{2\pi} \log \|a - x\|$. Compare the above representation to Cauchy's integral formula:

$$a = \frac{1}{2\pi i} \int_{\partial D} \frac{1}{z - a} \cdot z \, d\sigma_z$$

In 2D, Green and Complex Coordinates (Gotsman) are equivalent!

Derivation of Green Coordinates

- normal n_j constant on each triangle t_j
- for $x \in t_j$, $x = \sum_{v_k \in \mathbb{V}(t_j)} \Gamma_k(x) \cdot v_k$ (real barycentric coordinates; Γ_k piecewise linear hat function with $\Gamma_k(v_i) = \delta_{ik}$)

Rearrange and for $x = \sum_{i \in I_{\mathbb{V}}} \varphi_i(x) \cdot v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(x) \cdot n_j$, one obtains:

$$\varphi_i(a) = \int_{x \in \mathrm{AdjFaces}(v_i)} \Gamma_i(x) \cdot \frac{\partial G_a}{\partial n}(x)\, d\sigma_x \qquad i \in I_{\mathbb{V}}$$

$$\psi_j(a) = -\int_{x \in t_j} G_a(x)\, d\sigma_x \qquad j \in I_{\mathbb{T}}$$

Desired Properties

For the transformation

$$F(x, P') = \sum_{i \in I_{\mathbb{V}}} \varphi_i(x) \cdot v'_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(x) \cdot s_j \cdot n'_j,$$

the scaling factors s_j (depending on source and target cage!) are still to be defined to ensure the following properties:

1. Linear reproduction: $x = F(x, P)$
2. Translation invariance: $F(x, P + v) = x + v$
3. Rotation and scale invariance: $F(x, TP) = Tx$ for T an affine transformation consisting of rotation with isotropic scale
4. Shape preservation: $x \mapsto F(x, P')$ is conformal ($d = 2$) or quasi-conformal ($d \geq 3$)
5. Smoothness: ϕ_i, ψ_j should be smooth

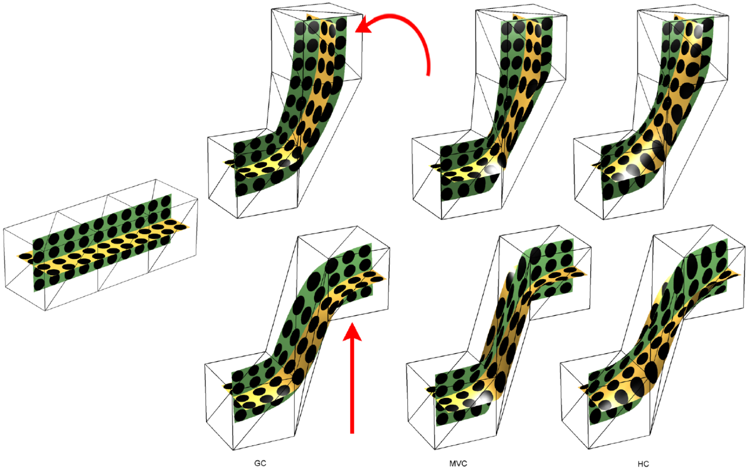
Scaling Factors

- In 2D, choose $s_j = \left\| t'_j \right\| / \left\| t_j \right\|$.
- In 3D, choose

$$\frac{1}{\sqrt{8}\text{area}(t_j)} \sqrt{\left\| u' \right\|^2 \left\| v \right\|^2 - 2(u' \cdot v')(u \cdot v) + \left\| v' \right\|^2 \left\| u \right\|^2},$$

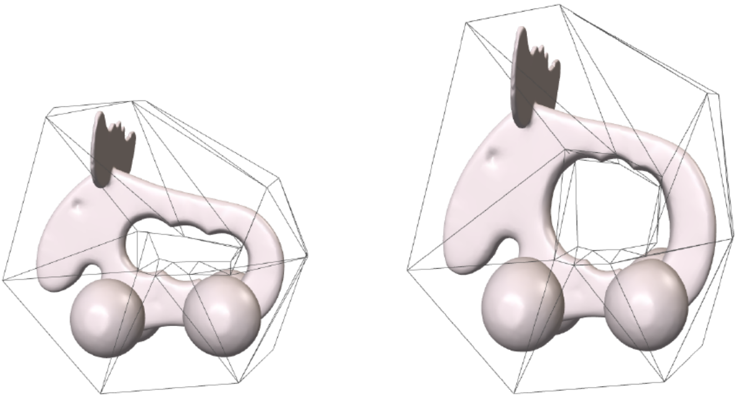
  where u, v, u', v' span the old and new triangles t_j, t'_j.
- If $t_j = t'_j$, then $s_j = 1$. (necessary for linear reproduction)
- Conformality for $d = 2$ is proven in Technical Report yet to be published.
- Quasi-Conformality for $d \geq 3$:
  - *distortion* measured by quotient of singular values of *DF*
  - experimentally found distortion bounded by constant ≤ 6 (Mean-Value Coordinates and Harmonic Coordinates yield unbounded distortion proportional to cage distortion)

Some Images I



Deformations using Green, Mean-Value, Harmonic Coordinates

Some Images II

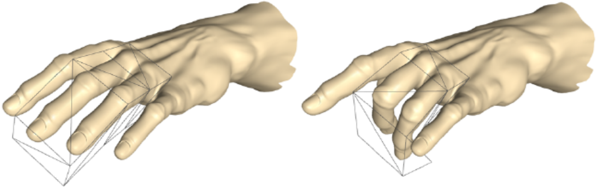


Deformation using a non-simply connected cage

Contents

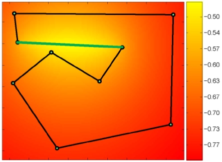
Partial Cages: Motivation

- Sometimes only part of a geometry should be deformed.
- Large cages are harder to construct and increase computation time.
- Requirements:
  - Smooth transition where geometry crosses “exit face”.
  - Diminishing influence of cage movement outside the cage.

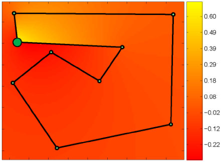


Problems

- Green's Identity only holds inside the cage, i. e. for $x \in P^{\text{in}}$.
- Coordinate functions:
  - Normal weights $\psi_j(a) = -\int_{x \in t_j} G_a(x)\, d\sigma_x$ are smooth across $\partial\Omega$:



  - Vertex weights $\varphi_i(a) = \int_{x \in \text{AdjFaces}(v_i)} \Gamma_i(x) \cdot \frac{\partial G_a}{\partial n}(x)\, d\sigma_x$ are discontinuous across adjacent faces of v_i:



- $F(x, P) = 0$ if $x \in P^{\text{ext}}$

Solution

- Goal: Find analytic (complex-analytic in $d = 2$, real-analytic in $d \geq 3$) continuations of φ_i across a fixed face t_r.
- Let $I_r \subset I_{\mathbb{V}}$ be the index set of vertices spanning t_r.
- Define $\tilde{\psi}_r$ and $\tilde{\varphi}_i$ ($i \in I_r$) such that:
  - linear reproduction holds:

$$\sum_{i \in I_r} \tilde{\varphi}_i(x) v_i + \tilde{\psi}_r(x) n_r = x - \sum_{i \in I_{\mathbb{V}} \setminus I_r} \varphi_i(x) v_i - \sum_{j \neq r} \psi_j(x) n_j$$

  - translation invariance holds:

$$\sum_{i \in I_r} \tilde{\varphi}_i(x) = 1 - \sum_{i \in I_{\mathbb{V}} \setminus I_r} \varphi_i(x)$$

  This yields an (invertible!) linear equation system that can be used to compute $\tilde{\varphi}_i(x)$ and $\tilde{\psi}_j(x)$.
- $\tilde{\varphi}_i(x) = \varphi_i(x)$ and $\tilde{\psi}_j(x) = \psi_j(x)$ if $x \in P^{\text{in}}$ (by construction)

Properties of Extension

Theorem

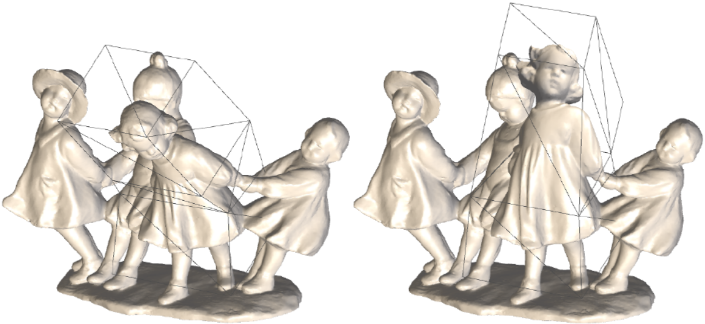
The mapping

$$\tilde{F}(x, P') = \sum_{i \in I_{\mathbb{V}}} \tilde{\varphi}_i(x) \cdot v'_i + \sum_{j \in I_{\mathbb{T}}} \tilde{\psi}_j(x) \cdot s_j \cdot n'_j$$

- *in the 2D case is the unique complex-analytic extension of the mapping $F(\cdot, P')$ through the edge t_r.*
- *In 3D, $\tilde{\varphi}_i$ and $\tilde{\psi}_j$ are the unique real-analytic extensions of φ_i, ψ_j through the face t_r.*

In some cases, it is possible to define an extension for multiple "exit faces".

Some Images



Deformation using a partial cage

Contents

Pseudocodes

Input: cage $P = (\mathbb{V}, \mathbb{T})$, set of points $\Lambda = \{\boldsymbol{\eta}\}$
Output: $2D$ GC $\phi_i(\boldsymbol{\eta}), \psi_j(\boldsymbol{\eta}), i \in I_{\mathbb{V}}, j \in I_{\mathbb{T}}, \boldsymbol{\eta} \in \Lambda$
`/* Initialization */`
set all $\phi_i = 0$ and $\psi_j = 0$
`/* Coordinate computation */`
foreach *point* $\boldsymbol{\eta} \in \Lambda$ **do**
  foreach *edge* $j \in I_{\mathbb{T}}$ *with vertices* $\boldsymbol{v}_{j_1}, \boldsymbol{v}_{j_2}$ **do**
    $\boldsymbol{a} := \boldsymbol{v}_{j_2} - \boldsymbol{v}_{j_1}$; $\boldsymbol{b} := \boldsymbol{v}_{j_1} - \boldsymbol{\eta}$
    $Q := \boldsymbol{a} \cdot \boldsymbol{a}$; $S := \boldsymbol{b} \cdot \boldsymbol{b}$; $R := 2\boldsymbol{a} \cdot \boldsymbol{b}$
    $BA := \boldsymbol{b} \cdot \|\boldsymbol{a}\| \boldsymbol{n}(t_j)$; $SRT := \sqrt{4SQ - R^2}$
    $L0 := \log(S)$; $L1 := \log(S + Q + R)$
    $A0 := \frac{tan^{-1}(R/SRT)}{SRT}$
    $A1 := \frac{tan^{-1}((2Q+R)/SRT)}{SRT}$
    $A10 := A1 - A0$; $L10 := L1 - L0$
    $\psi_j(\boldsymbol{\eta}) :=$
    $-\|\boldsymbol{a}\|/(4\pi) \left[\left(4S - \frac{R^2}{Q}\right) A10 + \frac{R}{2Q} L10 + L1 - 2 \right]$
    $\phi_{j_2}(\boldsymbol{\eta}) := \phi_{j_2}(\boldsymbol{\eta}) - \frac{BA}{2\pi} \left[\frac{L10}{2Q} - A10 \frac{R}{Q} \right]$
    $\phi_{j_1}(\boldsymbol{\eta}) := \phi_{j_1}(\boldsymbol{\eta}) + \frac{BA}{2\pi} \left[\frac{L10}{2Q} - A10 \left(2 + \frac{R}{Q}\right) \right]$
  end
end

Input: cage $P = (\mathbb{V}, \mathbb{T})$, set of points $\Lambda = \{\boldsymbol{\eta}\}$
Output: $3D$ GC $\phi_i(\boldsymbol{\eta}), \psi_j(\boldsymbol{\eta}), i \in I_{\mathbb{V}}, j \in I_{\mathbb{T}}, \boldsymbol{\eta} \in \Lambda$
`/* Initialization */`
set all $\phi_i = 0$ and $\psi_j = 0$
`/* Coordinate computation */`
foreach *point* $\boldsymbol{\eta} \in \Lambda$ **do**
  foreach *face* $j \in I_{\mathbb{T}}$ *with vertices* $\boldsymbol{v}_{j_1}, \boldsymbol{v}_{j_2}, \boldsymbol{v}_{j_3}$ **do**
    foreach $\ell = 1, 2, 3$ **do**
      $\boldsymbol{v}_{j_\ell} := \boldsymbol{v}_{j_\ell} - \boldsymbol{\eta}$
    $\boldsymbol{p} := (\boldsymbol{v}_{j_1} \cdot \boldsymbol{n}(t_j))\boldsymbol{n}(t_j)$
    foreach $\ell = 1, 2, 3$ **do**
      $s_\ell :=$
      $sign\left(((\boldsymbol{v}_{j_\ell} - \boldsymbol{p}) \times (\boldsymbol{v}_{j_{\ell+1}} - \boldsymbol{p})) \cdot \boldsymbol{n}(t_j)\right)$
      $I_\ell :=$ **GCTriInt**$(\boldsymbol{p}, \boldsymbol{v}_{j_\ell}, \boldsymbol{v}_{j_{\ell+1}}, 0)$
      $II_\ell :=$ **GCTriInt**$(0, \boldsymbol{v}_{j_{\ell+1}}, \boldsymbol{v}_{j_\ell}, 0)$
      $\boldsymbol{q}_\ell := \boldsymbol{v}_{j_{\ell+1}} \times \boldsymbol{v}_{j_\ell}$
      $\boldsymbol{N}_\ell := \boldsymbol{q}_\ell / \|\boldsymbol{q}_\ell\|$
    $I := -\left|\sum_{k=1}^{3} s_k I_k\right|$
    $\psi_j(\boldsymbol{\eta}) := -I$
    $\boldsymbol{w} := \boldsymbol{n}(t_j) I + \sum_{k=1}^{3} \boldsymbol{N}_k II_k$
    if $\|\boldsymbol{w}\| > \epsilon$ **then**
      foreach $\ell = 1, 2, 3$ **do**
        $\phi_{j_\ell}(\boldsymbol{\eta}) := \phi_{j_\ell}(\boldsymbol{\eta}) + \frac{\boldsymbol{N}_{\ell+1} \cdot \boldsymbol{w}}{\boldsymbol{N}_{\ell+1} \cdot \boldsymbol{v}_{j_\ell}}$
  end
end

Procedure **GCTriInt**$(\boldsymbol{p}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\eta})$
$\alpha := \cos^{-1}\left(\frac{(\boldsymbol{v}_2 - \boldsymbol{v}_1) \cdot (\boldsymbol{p} - \boldsymbol{v}_1)}{\|\boldsymbol{v}_2 - \boldsymbol{v}_1\| \|\boldsymbol{p} - \boldsymbol{v}_1\|}\right)$
$\beta := \cos^{-1}\left(\frac{(\boldsymbol{v}_1 - \boldsymbol{p}) \cdot (\boldsymbol{v}_2 - \boldsymbol{p})}{\|\boldsymbol{v}_1 - \boldsymbol{p}\| \|\boldsymbol{v}_2 - \boldsymbol{p}\|}\right)$
$\lambda := \|\boldsymbol{p} - \boldsymbol{v}_1\|^2 \sin(\alpha)^2$
$c := \|\boldsymbol{p} - \boldsymbol{\eta}\|^2$
foreach $\theta = \pi - \alpha, \pi - \alpha - \beta$ **do**
  $S := \sin(\theta)$; $C := \cos(\theta)$
  $I_\theta := \frac{-sign(S)}{2} \left[2\sqrt{c} \tan^{-1}\left(\frac{\sqrt{c}C}{\sqrt{\lambda + S^2 c}}\right) + \sqrt{\lambda} \log\left(\frac{2\sqrt{\lambda}S^2}{(1-C)^2}\left(1 - \frac{2cC}{c(1+C) + \lambda + \sqrt{\lambda^2 + \lambda c S^2}}\right)\right)\right]$
return $\frac{-1}{4\pi} \left| I_{\pi-\alpha} - I_{\pi-\alpha-\beta} - \sqrt{c}\beta \right|$

Pseudocode for 2D and 3D given in the paper

Complexity

- N number of geometry vertices, V number of cage vertices, T number of cage faces
- Preprocessing:
  - compute coordinates, $O(N \cdot (V + T))$ (but with large constants!)
- On every cage deformation:
  - compute new normals and scaling factors, $O(T)$
  - compute new positions, $O(N \cdot (V + T))$ (but can be done fast as simple matrix multiplication)
  - can be done more efficient: consider only changed vertices / triangles

For Further Reading

- Y. Lipman, D. Levin, D. Cohen-Or
  Green Coordinates.
  ACM SIGGRAPH 2008
- Y. Lipman, D. Levin
  On the derivation of green coordinates. *Technical Report*.
  unpublished