What is… a Geodesic on a Riemannian Manifold?

Notes and Overview

Tobias G. Pfeiffer[*] Dror Atariah[†]

Contents

1 Topological and Smooth Manifolds **1**

2 Riemannian Manifolds **4**

2.1 Tangent Space 4

2.2 Riemannian Manifold 6

3 Geodesics on Riemannian Manifolds **7**

Abstract

What does it mean to "go straight" on a sphere? What is the shortest distance between two points in a space other than the ordinary Euclidean $\mathbb{R}^n$? These two questions, and many more, are of geometrical nature, and are treated within the framework of differential geometry. The key object that is used is the *manifold*, which we will define in this talk. We will start from the broadest definition of a topological manifold, and end at the Riemannian one. Then we will give a basic idea and definitions of what a geodesic on a Riemannian manifold is, together with some examples.

1 Topological and Smooth Manifolds

The most general definition of a manifold:

Definition 1 (Topological Manifold)**.** *A topological space M is called a topological n-manifold if:*

- *M is Hausdorff (i.e. every pair p, q can be separated by two disjoint open sets) and second countable (i.e. there exists a countable basis to the topology of M).*
- *M is locally Euclidean, that is, for every $p \in M$ there exists an open set $U \subset M$ with $p \in U$ and an open set $V \subset \mathbb{R}^n$ such that there exists a homeomorphism $\varphi : U \to V$.*

A pair (U, φ), where $p \in U \subset M$ and φ is a homeomorphism, is called a *coordinate chart*. Here U is called *coordinate domain* and φ is the *coordinate map*. For every $q \in U$ the vector $\varphi(q) = (x^1(q), \dots, x^n(q)) \in \mathbb{R}^n$ is the *local coordinates of q in U*. The collection

$$\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$$

is called an *atlas* if $M = \bigcup_{\alpha \in I} U_\alpha$.

[*] tobias.pfeiffer@fu-berlin.de

[†] dror.atariah@fu-berlin.de

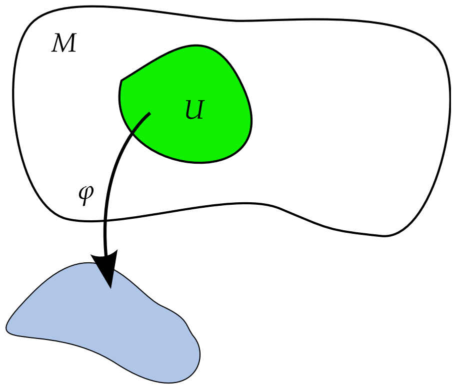


Figure 1: An example of a manifold with a chart (U, φ).

Given a topological manifold M, and two charts $(U, \varphi), (V, \psi)$, such that $U \cap V \neq \emptyset$. If the *transition map*

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

(cf. figure 2) is a diffeomorphism, (U, φ) and (V, ψ) are said to be *smoothly compatible*.

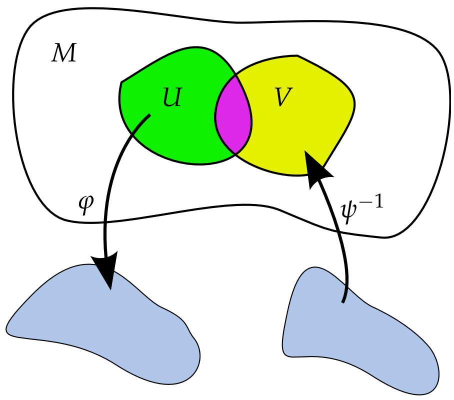


Figure 2: An example of a transition map $\varphi \circ \psi^{-1}$.

The atlas $\mathcal{A}$ is called *smooth atlas* if every two charts are smoothly compatible. Finally, we can define the notion of smooth manifold:

Definition 2 (Smooth Manifold)**.** *A topological n-manifold M is called a smooth manifold, if there exists a smooth atlas $\mathcal{A}$ of M.*

The above is derived from the first chapter of [Lee].

★ **Example: Sphere** The sphere $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\} \subset \mathbb{R}^3$ is a smooth 2-manifold.

- Since a subspace of a Hausdorff space is Hausdorff (refer to Theorem 31.2(a) in [Munkres, p.196]), then S^2 is also Hausdorff.
- As $\mathbb{R}^3$ is second countable (take as a basis the set of all products $(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ where a_i, b_i's are rational), then S^2 is also second countable as a subspace. See Theorem 30.2 in [Munkres, p. 191].
- Consider the coordinate domains of the form

$$U_i^+ = \{(x_1, x_2, x_3) \in S^2 \mid x_i > 0\}$$
$$U_i^- = \{(x_1, x_2, x_3) \in S^2 \mid x_i < 0\}$$

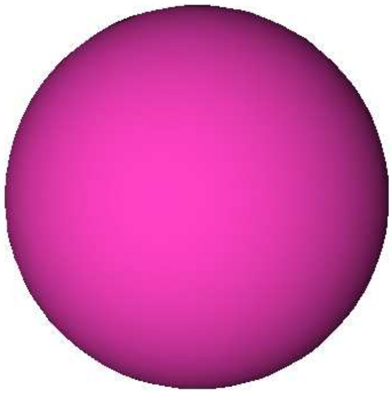


Figure 3: A unit sphere

That is, half-spheres in each direction. For each domain, let the homeomorphism be for example:

$$\varphi_1^{\pm}(x_1, x_2, x_3) = \pm(x_2, x_3)$$
$$(\varphi_1^{\pm})^{-1}(x_2, x_3) = (\pm\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3)$$

and so on for every i. As the domains $U_i^{\pm}$ covers the sphere we have an atlas. This atlas is also smooth. For example:

$$\varphi_2^{+} \circ (\varphi_1^{-})^{-1}(x, y) = \varphi_2^{+}(-\sqrt{1 - x^2 - y^2}, x, y) = (-\sqrt{1 - x^2 - y^2}, y).$$

Thus, S^2 is a smooth manifold. For more details, refer to examples 1.2 and 1.20 in [Lee].

★ **Example: Discrete Planar Curves** For a fixed $n \in \mathbb{N}$, we consider the set of discrete curves in $\mathbb{R}^2$ with n nodes:

$$M := \{c : [0, 1] \to \mathbb{R}^2 \mid c \text{ continuous, } c_{|[\frac{k-1}{n-1}, \frac{k}{n-1}]} \text{ linear for } k = 1, \ldots, n-1\}$$

as a subset of $C([0, 1], \mathbb{R}^2)$ (all curves in $\mathbb{R}^2$).

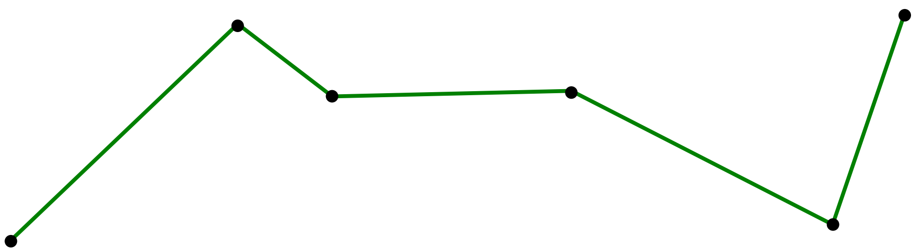


Figure 4: A discrete curve with $n = 6$ nodes.

By equipping M with the norm $\|\cdot\|_\infty$ (the standard norm for $C([0, 1], \mathbb{R}^2)$), we induce a metric $d(f, g) := \|f - g\|_\infty$ and thus a topology on M.

- M is Hausdorff: Every metric space is Hausdorff.
- M is second countable: M is a topological subspace of $C([0, 1], \mathbb{R}^2)$. From the Weierstraß approximation theorem, we know that the space of polynomials is dense in $C([0, 1], \mathbb{R}^2)$. Since the space of polynomials is second countable, so is M.
- M is locally Euclidean: Actually, M is globally Euclidean, because it can be directly identified with $\mathbb{R}^{2n}$ by specifying all the coordinates of the nodes of the curve:

$$\varphi : c \mapsto \big(c(0), c(\tfrac{1}{n-1}), \ldots, c(1)\big) \in \mathbb{R}^{2n}$$

The inverse mapping is

$$\varphi^{-1} : (x_1, y_1, \dots, x_n, y_n) \mapsto c, \quad c : t \mapsto (1 - \gamma(t)) \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \gamma(t) \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} \quad \text{for } t \in [\tfrac{k-1}{n-1}, \tfrac{k}{n-1}],$$

where $\gamma(t) := (n-1)t - k + 1$ is the affine map from $[\frac{k-1}{n-1}, \frac{k}{n-1}]$ to $[0, 1]$, i.e. c is the linear interpolant of the given nodes.

2 Riemannian Manifolds

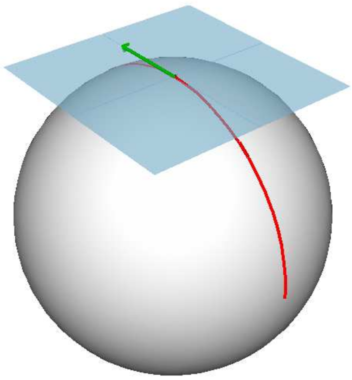
2.1 Tangent Space

In real analysis the derivative is a linear approximation of the function, so is the tangent vector of a planar or space curve. For a surface element, as the sphere, which is embedded in some $\mathbb{R}^n$, we can define tangent vectors and tangent space at a point in the following way:

★ **Example: Sphere** Let $p \in S^2$ be an arbitrary point and consider two smooth curves $c_1, c_2 : (-\varepsilon, \varepsilon) \to S^2$. We say that c_1 and c_2 are equivalent if:

- $c_1(0) = c_2(0) = p$
- $c_1'(0) = c_2'(0)$

This is clearly an equivalence relation on the set of all curves passing through p. We call each equivalence class a *tangent vector* of S^2 at p, and denote it by X_p. The set of all tangent vectors of S^2 at p is called the *tangent space* of S^2 at p.



In this case, the tangent space is nothing but a copy of $\mathbb{R}^2$.

Note that we can compute $c'(0)$ in this straightforward way only because of the embedding in $\mathbb{R}^3$, where we can use the tools of real analysis. In the general case of a smooth manifold, we can easily define a curve on M, and the derivative is defined as well, but not as easily.

We will now define a tangent vector to an arbitrary smooth manifold.

Definition 3 (Tangent Vector and Space)**.** *Given a smooth manifold M and a point p, then a* tangent vector *of M at p is an equivalence class of differentiable curves*

$$c : (-\varepsilon, \varepsilon) \to M$$

where $c(0) = p$ and $c_1 \sim c_2 \iff (\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$, i.e. the images of the curves under φ have the same tangent vector in $\mathbb{R}^n$. The tangent space *of M at p is the set of all tangent vectors at p.*

Remark 1 (Different definitions of the tangent vectors)**.** The definition given above is of geometrical nature. It is common to define the notion of the tangent vector in another two possible ways:

- Algebraic definition
- Physical definition

The algebraic definition is regarded as a *directional derivative* of scalar functions defined on the manifold in the following sense. If $c : I \to M$ is a smooth curve, then its tangent is:

$$c'(t_0) \in T_{c(t_0)}M, \quad t_0 \in I.$$

Now, given a smooth scalar function $f : M \to \mathbb{R}$, then the tangent vector $c'(t_0)$ can be thought of as a directional derivation of f at the point $c(t_0)$.

It can be shown that the tangent space is a vector space of dimension n: For every $v \in \mathbb{R}^n$, a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ can be easily constructed such that $\gamma'(0) = v$ and then the equivalence class of $c := \varphi^{-1} \circ \gamma$ under $\sim$ is a tangent vector corresponding to v. Using this construction, it can be seen that indeed $T_pM \cong \mathbb{R}^n$.

★ **Example: Discrete Planar Curves** Let $p \in M$, i.e. p is a discrete curve in $\mathbb{R}^2$ with n nodes. A curve $c : (-\varepsilon, \varepsilon) \to M$ through p is a set of planar curves with $c(0) = p$.

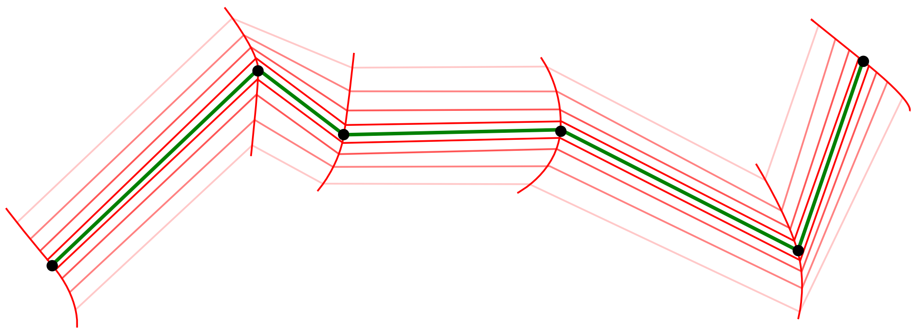


Figure 5: A curve c in M through p (which is itself a discrete curve in $\mathbb{R}^2$), shown in selected timesteps in $(-\varepsilon, \varepsilon)$.

Now we can't simply compute $c'(0)$, since we cannot compute $\frac{c(h)-c(0)}{h}$, as there is no addition/subtraction defined on M. However, we can look at the image of c under φ, which is an ordinary curve in $\mathbb{R}^{2n}$ and can compute $(\varphi \circ c)'(0) \in \mathbb{R}^{2n}$.

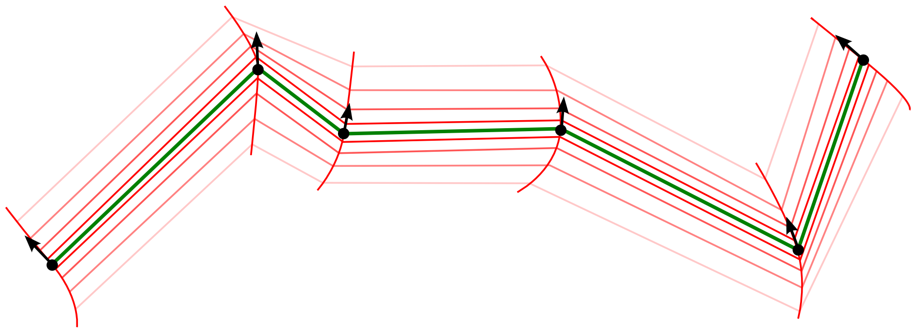


Figure 6: The arrows represent the components of $(\varphi \circ c)'(0)$ in $\mathbb{R}^{2n}$.

A tangent vector at p as defined above is the equivalence class of all curves on M through p whose image under φ has the same tangent vector in $\mathbb{R}^{2n}$, i.e. the tangent space is basically $\mathbb{R}^{2n}$ itself.

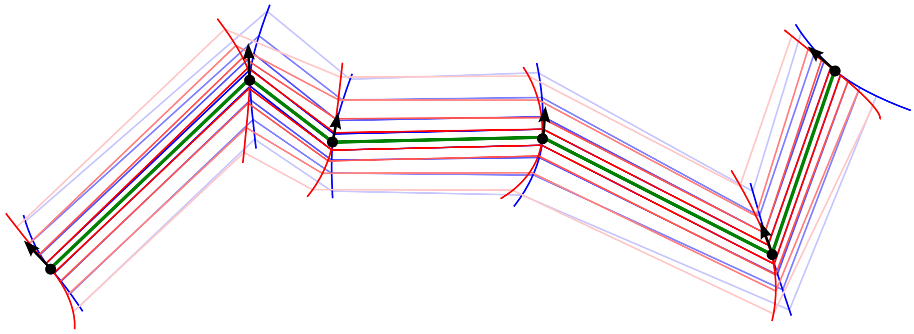


Figure 7: Two curves in M with the same tangent vectors when "pulled down" to $\mathbb{R}^{2n}$, i.e. belonging to the same equivalence class.

2.2 Riemannian Manifold

Definition 4 (Riemannian Metric)**.** *Let M be a smooth manifold. A Riemannian metric g on M is a differentiable map $p \mapsto g_p : T_pM \times T_pM \to \mathbb{R}$ such that g_p is*

- *bilinear: $g_p(aX_1 + bX_2, Y) = ag_p(X_1, Y) + bg_p(X_2, Y)$ (same in the second argument)*
- *symmetric: $g_p(X, Y) = g_p(Y, X)$*
- *positive definite: $g_p(X, X) > 0$ for $X \neq 0$.*

g_p should be thought of as a scalar product on T_pM; that is, a way to measure lengths and angles in the tangent space.

★ **Example: Sphere** The Riemannian metric g_p is inherited from the embedding space's scalar product, that is the Euclidean scalar product in $\mathbb{R}^3$:

$$g_p : T_pS^2 \times T_pS^2 \to \mathbb{R}$$
$$(X_p, Y_p) \mapsto g_p(X_p, Y_p) = \langle X_p, Y_p \rangle$$

★ **Example: Discrete Planar Curves** A Riemannian metric has to be defined on some representation of T_pM. If we identify T_pM with $\mathbb{R}^{2n}$, then the simplest possible Riemannian metric surely is the scalar product inherited from $\mathbb{R}^{2n}$:

$$g_p : T_pM \times T_pM \to \mathbb{R}, \quad (X, Y) \mapsto \langle X, Y \rangle$$

For a different representation of that same tangent space, in terms of tangent vectors being equivalence classes of curves on M, for two tangent vectors $[c_1]$ and $[c_2]$, we pick some representatives c_1^* and c_2^* of those equivalence classes and get the same metric through

$$g_p : T_pM \times T_pM \to \mathbb{R}, \quad ([c_1], [c_2]) \mapsto \langle (\varphi \circ c_1^*)'(0), (\varphi \circ c_2^*)'(0) \rangle$$

Definition 5 (Riemannian Manifold)**.** *The pair (M, g) is called a Riemannian manifold if M is a smooth manifold and g is a Riemannian metric.*

A Riemannian metric is not what is known as "metric" from the theory of metric spaces, but can be used to define a *distance function* with similar properties. As for the Euclidean scalar product, we can denote $\left|X_p\right|_g := \sqrt{g_p(X_p, X_p)}$ as the *length* of the tangent vector X_p and then talk about length of curves on manifolds:

Definition 6 (Length of a curve on a manifold)**.** *Let $c : [a, b] \to M$ be a piecewise smooth curve on M. Then*

$$L_g(c) := \int_a^b \left|c'(t)\right|_g \, dt \tag{1}$$

The length of a curve is independent of its parameterization, cf. [Lee, p. 275].

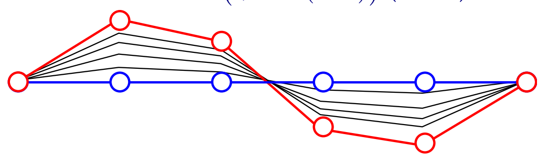
★ **Example: Sphere** Let S^2 be equipped with the standard metric from above and $c : [0, 2\pi] \to S^2, t \mapsto (\cos t, \sin t, 0)$ (i.e. c is the equator of S^2). Then

$$L_g(c) = \int_0^{2\pi} |c'(t)|_g \, \mathrm{d}t = \int_0^{2\pi} 1 \, \mathrm{d}t = 2\pi,$$

which is exactly what we would expect for the equator of the unit sphere.

★ **Example: Discrete Planar Curves** A curve in M is a set of discrete curves in $\mathbb{R}^2$. For $n = 6$, let $p, q \in M$ with $p(t) = (t, 0)$ (blue) and q the piecewise linear interpolant of the nodes $\big(t, \sin(2\pi t)\big)$ (red).

Then *one possible* curve connecting p and q is the curve $c : [0, 1] \to M$ where $c(s)$ is the piecewise linear interpolant of the nodes $\big(t, s\sin(2\pi t)\big)$ (black):



Then $(\varphi \circ c)(s) = \big(0, s\sin(0),\ \frac{1}{5}, s\sin(2\pi\frac{1}{5}),\ \frac{2}{5}, s\sin(2\pi\frac{2}{5}),\ \ldots,\ 1, s\sin(2\pi)\big) \in \mathbb{R}^{12}$
and $(\varphi \circ c)'(s) = \big(0, \sin(0),\ 0, \sin(2\pi\frac{1}{5}),\ 0, \sin(2\pi\frac{2}{5}),\ \ldots,\ 0, \sin(2\pi)\big) \in \mathbb{R}^{12}$

and with the definition of the curve length, we have that

$$\begin{aligned} L(c) &= \int_0^1 |c'|_g \, \mathrm{d}s = \int_0^1 \sqrt{g_p([c],[c])} \, \mathrm{d}s = \int_0^1 \sqrt{\langle (\varphi \circ c)'(s), (\varphi \circ c)'(s) \rangle} \, \mathrm{d}s \\ &= \int_0^1 \big|\big(0, \sin(0),\ \ldots,\ 0, \sin(2\pi)\big)\big|_{\mathbb{R}^{2n}} \, \mathrm{d}s = \int_0^1 \Big(\sum_{i=0}^{5} \sin^2(\tfrac{k}{5})\Big)^{1/2} \mathrm{d}s = \sqrt{\sum_{i=0}^{5} \sin^2(\tfrac{k}{5})} \end{aligned}$$

Using the curve length definition from above, we can define the distance between two points in M.

Definition 7 (Intrinsic distance of points on a manifold)**.** *Let (M, g) be a connected Riemannian manifold and $p, q \in M$. Then we can define the* intrinsic distance *or* geodesic distance *between p and q as*

$$d_g(p, q) := \inf\{L_g(c) \mid c : [0, 1] \to M \textit{ piecewise smooth},\ c(0) = p,\ c(1) = q\},$$

i.e. the length of the shortest curve in M connecting p and q.

The geodesic distance d_g indeed fulfills the axioms for a metric (in the sense of metric spaces) [Lee, p. 277], i.e.

- $d_g(x, y) \geq 0$ (and $d_g(x, y) = 0 \Leftrightarrow x = y$)
- $d_g(x, y) = d_g(y, x)$
- $d_g(x, y) \leq d_g(x, z) + d_g(z, y)$

3 Geodesics on Riemannian Manifolds

In the Euclidean space, going "straight" from a point p to a point q is very intuitive: It is nothing but going along the line segment connecting the two end points. In this case, the *straight* line is also the *shortest* path in the manifold connecting these points. A physical intuition of straight would be: *"Use* *nothing but the initial velocity!"*. Just like a bullet being fired from a gun (when neglecting gravitational force, wind, etc.), or motorcycle rider who does not turn the handlebars. Roughly speaking, this means that the velocity vector does not change in any tangential direction but its own. More formally, this is nothing but saying that the directional derivative of the velocity *in its own* direction vanishes.

For an example, refer to the Motorcycle clip.

In what follows, we will give an abstract definition which should be thought of as a directional derivative. First we have to define the notion of a differentiable vector field; this field will associate to every point of the manifold a tangential vector in a differentiable manner.

Definition 8 (Tangential Vector Field). *Given a smooth manifold M, then the map X that assigns a tangent vector $X_p \in T_pM$ for every point $p \in M$ is called a differentiable vector field on M, if for every chart $\varphi : U \to V \subset \mathbb{R}^n$ where $p \in U$, then the coefficients $\xi^i : U \to \mathbb{R}$:*

$$X_p = \sum \xi^i(p) b_i^{\varphi}(p)$$

are differentiable functions. The set $\{b_i^{\varphi}(p)\}$ is a basis of the tangent space T_pM depending on the coordinate system derived from the chart.

In what follows we will refer to tangential vector field simply as *vector field*. The last definition we are still missing is of the so called *Lie bracket*.

Definition 9 (Lie Bracket). *Given two vector fields X, Y on M, then the Lie bracket is the vector field $[X, Y]$ which satisfies:*

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

where $f : M \to \mathbb{R}$. $X(f)$ and $Y(f)$ are the directional derivatives of f in direction X and Y respectively.

Now, we are ready to define the directional derivative of vector fields.

Definition 10 (Riemannian Connection). *Let (M, g) be a Riemannian manifold and X, Y be two vector fields. The Riemannian connection is the map $\nabla(X, Y) = \nabla_X Y$ that maps the pair (X, Y) to a third vector field, such that the following properties hold:*

1. $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$
2. $\nabla_{fX} Y = f \nabla_X Y$
3. $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
4. $\nabla_X (fY) = f \nabla_X Y + \nabla_X f Y$
5. $\nabla_X (g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
6. $\nabla_X Y - \nabla_Y X - [X, Y] = 0$

Theorem 1. *Given a Riemannian manifold (M, g), then the Riemannian connection is uniquely determined.*

Remark 2. If M is a surface element, i.e. a 2-manifold in $\mathbb{R}^3$ with the Euclidean scalar product as its Riemannian metric, then the Riemannian connection is:

$$\nabla_X Y = D_X Y - \langle D_X Y, N \rangle N \tag{2}$$

where N is the normal of the surface. This holds only because the manifold (surface) is embedded in $\mathbb{R}^3$ and the notion of *normal* is well defined. In general we cannot utilize this definition as it is not intrinsic.

Finally, we can define the notion of a geodesic.

Definition 11 (Geodesic Curve). *An arc length parameterized curve $c : I \to M$ is called a geodesic if*

$$\nabla_{c'} c' = 0$$

Remark 3. If we consider a curve $c : I \to M$ where M is a surface, then the connection given in equation (2) vanishes whenever the curve's tangent does not change in any tangential direction.

Remark 4 (Initial Value Problem)**.** Let $p \in M$ be a point on a 2-manifold embedded in $\mathbb{R}^3$, and let $X \in T_pM$ be a tangent vector based at p. Now, consider the following initial value problem of finding a smooth curve $\gamma : I \to M$ such that:

$$\begin{aligned} \nabla_{\gamma'}\gamma' &= 0 \\ \gamma(0) &= p \\ \gamma'(0) &= X \end{aligned}$$

Using the Riemannian connection defined in equation (2), we have that the problem stated above is a 2$^{\text{nd}}$ order ODE and thus it has a unique solution for some open interval $(-\varepsilon, \varepsilon)$.

This initial value problem can be stated in the abstract setting as well, namely, for every point $p \in M$ and a tangent vector $X \in T_pM$ with $g_p(X, X) = 1$, then there exists an $\varepsilon > 0$ and a uniquely determined geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ which is arc length parameterized and $\gamma(0) = p$ and $\gamma'(0) = X$. For more details, refer to corollary 5.18(iii) in [Kuehnel].

Geodesics and Shortest Paths Theorem 4.13 in [Kuehnel] states that a shortest curve between two points on a manifold is always a geodesic. This is an amazing property that makes geodesics also interesting for a number of minimization problems.

On the other hand, locally, geodesics are distance minimizers. That is, for two "close" points along a geodesic γ, say $p = \gamma(t_0)$ and $q = \gamma(t_1)$, then the following holds:

$$d_g(p, q) = L_g(\gamma_{|[t_0, t_1]}).$$

For an example, refer to the Geodesics on a Bretzel clip.

★ **Example: Sphere** On a sphere, geodesics are always parts of great circles, i.e. equator-like curves. However, in general there are two geodesics connecting two points $p, q \in S^2$ (if $p = -q$, there are infinitely many), as can be seen in figure 8: One that is the shortest connection and one that is "the rest of the great circle".

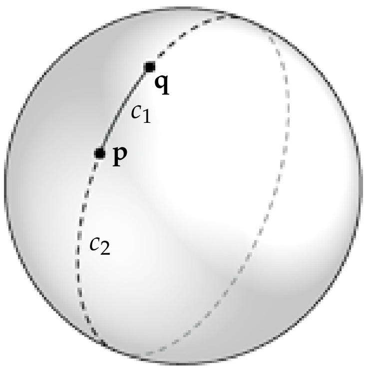


Figure 8: Two geodesics connecting p and q on a sphere.

★ **Example: Discrete Planar Curves** With the Riemannian metric that we have chosen above, i.e. just inheriting the scalar product from $\mathbb{R}^{2n}$, the shortest curves in M are equivalent to straight lines in $\mathbb{R}^{2n}$. That means that if the curve c connecting $p, q \in M$ just moves every point of the discrete curve p linearly to the corresponding point of the discrete curve q, c is a geodesic.

References

[Kuehnel] Wolfgang Kühnel. *Differentialgeometrie. Kurven, Flächen, Mannigfaltigkeiten.* Vieweg Studium: Aufbaukurs Mathematik, 2008.

[Lee] John M. Lee. *Introduction to Smooth Manifolds*. Springer, 2002.

[Munkres] James R. Munkres. *Topology*. Pearson Prentice Hall, 2000.